

# The 2+1 Dimensional NJL Model at Finite Temperature

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## Abstract

We describe properties of 2 + 1-dimensional Nambu-Jona-Lasinio (NJL) models at finite-temperature, beginning with the model with a discrete chiral symmetry. We then consider the model with a continuous  $U(1) \times U(1)$  chiral symmetry, describing the restoration of the symmetry at finite temperature. In each case, we compute the free energy and comment on a recently proposed constraint based upon it. We conclude with a brief discussion of NJL models with larger chiral symmetries.

## 1 Introduction

The behavior of QCD at finite temperature and density has received renewed attention recently with the coming of the RHIC experimental program. The study of this phase structure is difficult for any strongly coupled quantum field theory, and it can be helpful to examine the problem in nontrivial but tractable models. In particular, theories in less than four space-time dimensions can offer interesting and complex behavior as well as tractability,

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often in the form of large- $N$  expansions. In the case of three space-time dimensions, they can even be directly physical, describing various planar condensed matter systems. Models of interest for the study of finite temperature *and* density involve fermionic degrees of freedom and fall into two broad classes. One utilizes four-fermion interactions of the Nambu-Jona-Lasinio (NJL), or Gross-Neveu, form, and the other includes three dimensional gauge theories and closely related Thirring models.

We restrict our attention in this paper to NJL models at finite temperature  $T$ . The features we discuss are interesting in their own right and will play an important role in understanding temperature-density phase diagrams for these models. All the analysis will make use of a  $1/N_f$  expansion where  $N_f$  is the number of fermion species. The theories under consideration are renormalizable in the  $1/N_f$  expansion unlike in the loop expansion [2]. The expansion is reliable for all temperatures  $T$  except those in the vicinity of a certain critical temperature  $T_c$ .

The question of interest is the spontaneous breaking of chiral symmetry. Although this is in some ways easier to handle in three space-time dimensions ( $3D$ ) than in four, there is an important subtlety in the case of *continuous* symmetry. The Coleman-Mermin-Wagner (CMW) theorem stipulates that the spontaneous breaking of a continuous symmetry can not happen in  $3D$  at finite  $T$  [1]. This statement is related to the fact that it is impossible to write down a consistent theory of massless scalars in  $2D$ . If a spontaneous breaking of continuous symmetry were to happen at finite  $T$ , then one would be faced with this problem at momentum scales below  $T$ , i.e. it would be impossible to construct an effective  $2D$  theory of the Goldstone bosons' zero-modes.

We first review the behavior of the  $3D$  NJL model with a discrete chiral symmetry, where the CMW problem does not arise. The broken symmetry at  $T = 0$  remains broken at finite  $T$  up to a critical value  $T_c$ . We then turn to the  $3D$  NJL model with a continuous  $U(1) \times U(1)$  chiral symmetry, exploring symmetry restoration at all  $T > 0$ . A critical temperature  $T_c$

continuous to exist, but it now marks a transition from the ordinary symmetric phase at high temperature to a low temperature Kosterlitz-Thouless phase that is also chirally symmetric. The broken phase exists only at  $T = 0$ .

For each model, we also compute the thermodynamic free energy and enumerate the thermodynamic degrees of freedom. In a recent paper [3], the free energy  $F(T)$  was used as the basis for a proposed constraint on the behavior of asymptotically free theories. It was observed that for any theory governed by a fixed point in the ultraviolet (UV) or infrared (IR), the dimensionless quantity  $f(T) \equiv -2\pi[F(T) - F(0)]/\xi(3)T^3$  approaches a finite value in the corresponding limit (  $f_{UV} = f(T \rightarrow \infty)$  and  $f_{IR} = f(T \rightarrow 0)$  ), and counts the (effectively massless) degrees of freedom if the fixed point is trivial. It was noted that  $f_{UV} \geq f_{IR}$  for all known asymptotically free theories in which the IR behavior is also free, or weakly interacting, allowing  $f_{IR}$  to be computed. It was conjectured that this is true for all asymptotically free theories. The theories considered in this paper are governed by nontrivial UV fixed points and apriori may or may not satisfy this condition.

The outline of the paper is as follows: in Section 2, we review the NJL model with a discrete chiral symmetry; in Section 3, we treat the NJL model with a continuous  $U(1) \times U(1)$  chiral symmetry; we summarize and briefly describe NJL models with larger chiral symmetries in Section 4.

## 2 Discrete Chiral Symmetry

The NJL model with a discrete chiral symmetry for  $N_f$  copies of Dirac fermions is described by the following Lagrangian<sup>1</sup>:

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$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi + \frac{g_0^2}{2N_f} (\bar{\Psi}\Psi)^2, \quad (1)$$

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<sup>1</sup>We adopt the notation of [2] for  $\gamma$ -matrices:  $\gamma^\mu = \sigma^\mu \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$  and  $i\gamma_5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$  in (2+1)-dimensional Minkowski space with  $(1, -1, -1)$  signature.

where  $\Psi$  may be taken to be a  $4N_f$ -component fermion field. The discrete symmetry is  $\Psi \rightarrow \gamma_5 \Psi$ , and  $g_0$  is the coupling. We analyze the model using the  $1/N_f$  expansion.

We work first to leading order and then note that for this discrete-symmetry model the qualitative behavior is not modified in higher orders. It is convenient to introduce an auxiliary field  $\sigma$  coupled to  $\bar{\Psi}\Psi$ . The finite temperature effective potential as a function of  $\sigma$  may be computed to this order by integrating out the fermions:

$$V_{eff}(\sigma)/N_f = \frac{\sigma^2}{2g_0^2} + i T \text{Tr} \ln(i\cancel{\partial} - \sigma), \quad (2)$$

with the second term evaluated at finite temperature.

The only cutoff dependence is in the renormalization of  $g_0$ , and it may be carried out at zero temperature. Extremizing the zero-temperature effective potential ( $\partial V_{eff}/\partial\sigma = 0$ ) gives

$$\frac{1}{g_0^2} = 4 \int^\Lambda \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + \sigma_0^2} = \frac{2}{\pi^2} \left( \Lambda - \sigma_0 \tan^{-1} \frac{\Lambda}{\sigma_0} \right) \quad (3)$$

where  $\Lambda$  is an ultraviolet cutoff and where the extremal value  $\sigma_0$  describes the fermion mass gap at zero temperature. As  $g_0^2 \Lambda$  approaches the critical value  $\pi^2/2$  from above,  $\sigma_0/\Lambda \rightarrow 0$ . Equivalently, the cutoff may be removed holding  $\sigma_0$  fixed providing that  $g_0^2 \Lambda$  is tuned to  $\pi^2/2$ . With this prescription, it can be seen that the high energy behavior of the theory is described by a nontrivial but weak ultraviolet fixed point. That is, a finite effective four-fermion interaction  $\bar{g}^2(p)/2N_f$  is induced such that for  $p \gg \sigma_0$ , the dimensionless quantity  $\bar{g}^2(p) \cdot p \rightarrow \mathcal{O}(1)$ .

To leading order in  $1/N_f$ , the zero-temperature effective potential is now

$$V_{eff}^{T=0}(\sigma)/N_f = \frac{1}{3\pi} \sigma^3 - \frac{\sigma_0}{2\pi} \sigma^2, \quad (4)$$

which has a minimum at  $\sigma = \sigma_0$  and is convex for  $\sigma > \sigma_0/2$ . At finite temperature, the leading order effective potential takes the form:

$$V_{eff}(\sigma)/N_f = \frac{1}{3\pi} \sigma^3 - \frac{\sigma_0}{2\pi} \sigma^2 - T^3 \int_0^\infty \frac{dx}{\pi} \ln \left( 1 + \exp \left[ -\sqrt{x + \frac{\sigma^2}{T^2}} \right] \right). \quad (5)$$

It may be extremized to give the temperature dependent gap equation

$$\sigma_T = \sigma_0 - 2T \ln\left(1 + \exp[-\frac{\sigma_T}{T}]\right) = \sigma_0 - 2T \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \exp[-k\frac{\sigma_T}{T}] . \quad (6)$$

This leads to the critical temperature  $T_c = \sigma_0/(2 \ln 2)$ , above which  $\sigma_T$  vanishes. The existence of a finite  $T_c$  for a discrete symmetry in  $2 + 1$  dimensions is perfectly acceptable and this qualitative feature is not changed by the higher order terms in the  $1/N_f$  expansion.

Before discussing higher order terms, we comment on the thermodynamic free energy of this model which is given by  $F(T) = V_{eff}(\sigma_T)$ . Then, to leading order in  $1/N_f$ , the  $f(T)$  defined in the introduction will take the form

$$f(T) = - \frac{2N_f}{\zeta(3)T^3} \left( \frac{\sigma_T^3}{3} - \frac{\sigma_0}{2} \sigma_T^2 + \frac{\sigma_0^3}{6} + 2T^3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \left(1 + k\frac{\sigma_T}{T}\right) \exp[-k\frac{\sigma_T}{T}] \right) , \quad (7)$$

where  $\sigma_T$  is determined by the gap equation (6). It can be shown that this function increases monotonically from  $f_{IR} = 0$  at  $T = 0$  to  $f_{UV} = 3N_f$  at  $T = \infty$ . The infrared value,  $f_{IR} = 0$ , is expected because a mass gap develops for  $T < T_c$ . The ultraviolet value,  $f_{UV} = 3N_f$ , counts the number of fermionic degrees of freedom ( $4 N_f$ ) times the Fermi-Dirac factor ( $3/4$ ). This is expected because the model is governed in the ultraviolet by a weak UV fixed point – there will be higher order corrections in the  $1/N_f$  expansion. Monotonicity may be established by noting that

$$\begin{aligned} \frac{df(\sigma_T, T)}{dT} &= \frac{\partial f}{\partial \sigma_T} \frac{\partial \sigma_T}{\partial T} + \frac{\partial f}{\partial T} = \\ &= \frac{1}{\zeta(3)T^4} \left( \sigma_T^3 - \frac{3\sigma_0}{2} \sigma_T^2 + \frac{\sigma_0^3}{2} \right) + \frac{2}{\zeta(3)} \frac{\sigma_T^2}{T^3} \ln\left(1 + \exp[-\frac{\sigma_T}{T}]\right) . \end{aligned} \quad (8)$$

The first term is always non-negative since  $\sigma_T \leq \sigma_0$ ; the second term is always positive.

Finally we describe the effect of the higher order terms in the  $1/N_f$  expansion. Through next order, the effective potential is given by

$$V_{eff}(\sigma)/N_f = \frac{\sigma^2}{2g_0^2} + iTr \ln(i\partial - \sigma) + \frac{i}{2N_f} Tr \ln D_\sigma^{-1} , \quad (9)$$

where  $D_\sigma^{-1}(p^2)$  is the inverse  $\sigma$  propagator computed in a background  $\sigma$  field at finite temperature  $T$ . At zero temperature, this propagator is given by

$$D_\sigma^{-1}(p^2) = N_f \left( \frac{1}{g_0^2} - i \int \frac{d^3 q}{(2\pi)^3} Tr \frac{1}{q - \sigma} \frac{1}{q - p - \sigma} \right), \quad (10)$$

Using the leading order gap equation (3) and taking the limit  $\Lambda \rightarrow \infty$  with  $\sigma_0$  fixed,  $D_\sigma^{-1}(p^2)$  takes the form

$$D_\sigma^{-1}(p^2) = \frac{N_f}{\pi} \left[ \sigma - \sigma_0 + \frac{p^2 + 4\sigma^2}{2\sqrt{p^2}} \tan^{-1} \frac{\sqrt{p^2}}{2\sigma} \right]. \quad (11)$$

For nonzero temperature  $D_\sigma^{-1}(p^2)$  is given in Ref. [4]. Expressions, similar to Eq. 9 may be written down for higher order terms.

For  $T$  well above  $T_c$ , the  $1/N_f$  expansion for  $V_{eff}(\sigma)$  may be seen to converge for large  $N_f$ . As  $T \rightarrow T_c$  from above, however,  $m^2 \sim T - T_c$ . In this limit, higher order terms in  $V_{eff}(\sigma, \pi)$  become singular and trigger the breakdown of the  $1/N_f$  expansion. They may be described by an effective 2D, Landau-Ginzburg theory, consisting of the zero mode of the  $\sigma$  field, relevant at scales below  $T$ . The terms in the Landau-Ginzburg Lagrangian, in addition to the common mass term  $m^2\sigma^2/2$ , are a kinetic term, and a  $\lambda\sigma^4/4!$  interaction term. Each may be computed to any order in  $1/N_f$  by integrating out the fermions. To leading order, each arises with coefficient  $N_f$ .

Using this effective 2D theory and counting infrared powers, it may be seen that the effective dimensionless expansion parameter for  $V_{eff}(\sigma)$ , for small  $\sigma$ , is of the form

$$\frac{1}{N_f} \frac{T^2}{M^2}, \quad (12)$$

where  $M^2 = m^2 + \lambda\sigma^2$ , up to logarithmic corrections. Thus in the neighborhood of the origin, the  $1/N_f$  expansion breaks down for  $|T - T_c|/T \lesssim 1/N_f$ . In particular, it breaks down for the free energy and  $f(T)$ . This is true also as  $T$  approaches  $T_c$  from below, as indicated by the absolute value sign [6].

The  $1/N_f$  expansion remains convergent as long as  $|T_c - T|/T \gg 1/N_f$ . In the high temperature limit, it correctly describes the effective potential and free energy in the sym-

metric phase, leading to small corrections to the leading order estimate  $f_{UV} = 3N_f$ . In the low temperature limit, it describes the broken phase, and because of the mass gap, leads to  $f_{IR} = 0$  to all orders. The inequality  $f_{UV} \geq f_{IR}$  is satisfied.

### 3 Continuous $U(1) \times U(1)$ Chiral Symmetry

The NJL model with a continuous  $U(1) \times U(1)$  symmetry  $\Psi \rightarrow \exp(i\alpha\gamma_5)\Psi$ ,  $\Psi \rightarrow \exp(i\beta)\Psi$  is described by the Lagrangian [2]:

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi + \frac{g_0^2}{2N_f} \left[ (\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2 \right]. \quad (13)$$

The model may be analyzed to leading order in the  $1/N_f$  expansion by introducing auxiliary fields  $\sigma$  and  $\pi$  and integrating out the fermions. The leading-order, finite-temperature effective potential takes the form

$$V_{eff}(\sigma, \pi)/N_f = \frac{\sigma^2 + \pi^2}{2g_0^2} + i \text{Tr} \ln \left( i\not{\partial} - \sigma - i\gamma_5\pi \right) \quad (14)$$

As in the case of discrete symmetry, the renormalization may be performed by extremizing the effective potential ( $\partial V_{eff}/\partial\sigma = 0$  and  $\partial V_{eff}/\partial\pi = 0$ ) at zero temperature. The vacuum expectation values may be rotated using the chiral symmetry so that  $\pi$  has a vanishing expectation value. The zero temperature gap equation then takes the same form as in the discrete-symmetry model:

$$1 = \frac{2}{\pi^2} \left( g_0^2 \Lambda \right) \left( 1 - \frac{\sigma_0}{\Lambda} \tan^{-1} \frac{\Lambda}{\sigma_0} \right), \quad (15)$$

where  $\Lambda$  is an ultraviolet cutoff. As before,  $\sigma_0$  may be held fixed in the “continuum” limit  $\Lambda \rightarrow \infty$  ( $g_0^2\Lambda \rightarrow \pi^2/2$ ) describing the fermion mass gap at zero temperature. The chiral symmetry is broken and the  $\pi$  field describes the associated Goldstone boson. The zero temperature effective potential to leading order in  $1/N_f$  now reads:

$$V_{eff}(\sigma, \pi)/N_f = \frac{1}{3\pi} \left( \sigma^2 + \pi^2 \right)^{3/2} - \frac{\sigma_0}{2\pi} \left( \sigma^2 + \pi^2 \right), \quad (16)$$

which has degenerate minima at  $\sigma^2 + \pi^2 = \sigma_0^2$  and is convex as a function of two variables if  $\sigma^2 + \pi^2 \geq \sigma_0^2$ .

At finite temperature, the leading order effective potential has the same form as Eq. (5), with the replacement  $\sigma^2 \rightarrow \sigma^2 + \pi^2$ :

$$V_{eff}(\sigma, \pi)/N_f = \frac{1}{3\pi} (\sigma^2 + \pi^2)^{3/2} - \frac{\sigma_0}{2\pi} (\sigma^2 + \pi^2) - T^3 \int_0^\infty \frac{dx}{\pi} \ln \left( 1 + \exp \left[ -\sqrt{x + \frac{(\sigma^2 + \pi^2)}{T^2}} \right] \right). \quad (17)$$

Extremizing it suggests that, as in the case of discrete symmetry, the broken symmetry of the zero temperature theory remains broken at finite temperatures below a critical value  $T_c = \sigma_0/2 \ln 2$ . But unlike the discrete case, this conclusion cannot be correct since the zero-mode of the associated Goldstone boson would describe an effective 2D theory with a spontaneously broken continuous symmetry – in contradiction with the Coleman-Mermin-Wagner theorem.

We explore the resolution of this problem by first noting that as in the case of discrete symmetry, the  $1/N_f$  expansion is convergent as long as  $T$  is not near the transition temperature  $T_c$  ( $|T_c - T|/T \gg 1/N_f$ ). The next order term in  $V_{eff}(\sigma, \pi)/N_f$ , for example, may be written in the form

$$\frac{i}{2N_f} \text{Tr} \ln D_\sigma^{-1} + \frac{i}{2N_f} \text{Tr} \ln D_\pi^{-1}, \quad (18)$$

where  $D_\sigma^{-1}$  and  $D_\pi^{-1}$  are functions of  $T$ ,  $T_c$ , momentum, and field strength [4]. This term can be seen to be of order  $1/N_f$  for  $|T_c - T|/T \gg 1/N_f$ . But as in the case of discrete symmetry, the  $1/N_f$  expansion breaks down due to infrared singularities when  $|T - T_c|/T \lesssim 1/N_f$ , with the singular terms describable by an effective 2D Landau-Ginzburg Lagrangian.

For  $T \gg T_c$ , the theory is in the symmetric phase with a convergent  $1/N_f$  expansion. For  $T \ll T_c$ , even though the  $1/N_f$  expansion is again convergent, the continuous symmetry model is, unlike the discrete symmetry model, not in the broken phase. The chiral symmetry remains unbroken, although in a way different than in the high temperature range. To see

this, it is convenient to use the following parametrization<sup>2</sup> of the auxiliary fields for  $T < T_c$  [5]:

$$\sigma + i\pi = \rho e^{i\theta} . \quad (19)$$

The leading order potential, Eq. 17, then depends only on  $\rho$ , and extremization leads to a non-zero VEV  $\rho_T$ , equal to  $\rho_0 \equiv \sigma_0$  at  $T = 0$  and vanishing like  $(T_c - T)^{1/2}$  as  $T \rightarrow T_c$ . The fermion has mass  $\rho_T$ , the fluctuations of  $\rho$  have mass  $2\rho_T$  and there is a massless scalar field  $\theta$ .

This behavior implies symmetry breaking, however, only if  $\theta$  takes on some fixed and non-zero vacuum value. Now as is well known, this doesn't happen in 2D since quantum effects generate logarithmically infrared divergent fluctuations. In the present model at finite  $T$ , the same is true since the long distance behavior is effectively 2D. To explore this in detail, we first restrict attention to  $T$  sufficiently below  $T_c$  ( $(T_c - T)/T \gg 1/N_f$ ) so that the  $1/N_f$  expansion is a reliable tool. The realization of the symmetry may be studied by examining the theory in the infrared, at momentum scales well below  $\rho_T$ , where the fermions and the  $\rho$  fluctuations may be integrated out. The only massless degree of freedom is  $\theta$ , and if  $\rho_T$  and  $T$  are of the same order, the non-zero Matsubara frequencies of  $\theta$  may also be integrated out. The effective low energy theory describing physics at momentum scales well below  $T$  and  $\rho_T$  is then a 2D chiral Lagrangian [6, 7]:

$$L_{eff} = \frac{1}{2t} (\partial\theta)^2 + \dots , \quad (20)$$

where the dots indicate higher derivative terms in  $\theta$ . The dimensionless coefficient  $1/t$  is given to leading order in the  $1/N_f$  expansion by

$$\frac{1}{t} = N_f \frac{\rho_T}{4\pi T} \tanh\left(\frac{\rho_T}{2T}\right) . \quad (21)$$

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<sup>2</sup> This transformation of variables must be handled carefully. The jacobian of the transformation in the functional integral leads to a cubically divergent term in the effective lagrangian of the form  $\delta^3(0) \ln \rho$ , which serves only to cancel other such terms arising in the quantum computation [9]. Without this cancellation, the jacobian term might be assumed [10] to change the cutoff dependence from that described in Eq. 15, eliminating the existence of the continuum limit as described there.

In the very low temperature limit ( $T \ll T_c$ ), physics *at* momentum scales of order  $T$  as well as below it may be described by an effective theory obtained by integrating out the fermions and the  $\rho$  fluctuations, but keeping all the Matsubara frequencies of  $\theta$ . This theory is described by the effective chiral Lagrangian (20), but in three dimensions rather than two.

Since the underlying theory is Abelian, this effective Lagrangian describes a free massless scalar field  $\theta$ . Chiral symmetry breaking depends on the behavior of correlation functions of physical, and therefore single valued functions of  $\theta$  ( $\cos \theta$ ,  $\sin \theta$ , or equivalently  $e^{\pm i\theta}$ ). This behavior is determined by the coefficient  $t$  in the chiral Lagrangian, which is of order  $1/N_f$  or smaller unless  $\rho_T/T$  is small. (This only happens if  $T$  approaches  $T_c$  and we are avoiding that limit now to insure convergence of the  $1/N_f$  expansion). Analysis of this 2D theory, including the role of vortex solutions [7], reveals that for small  $t$ , it is in the Kosterlitz-Thouless [8] phase where the contribution from vortices is negligible, and where, for example, the correlation function  $\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle$  has the characteristic power-law behavior  $\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim x^{-t/2\pi}$  at distances  $x \gg 1/T$ . The power-law fall-off, while corresponding to an infinite correlation length, still indicates an absence of long range order. There is no chiral symmetry breaking.

The transition from broken to unbroken chiral symmetry is at  $T = 0$ . In the limit  $T \rightarrow 0$  the parameter  $t \rightarrow 0$ , and the range of relevance for the effective 2D theory and the (weakening) power-law fall off moves off to infinity. In the zero temperature 3D theory, the  $\theta$  field develops a fixed VEV and describes a Goldstone boson.

At finite  $T$ , the analysis of the effective Abelian 2D theory leads to the conclusion [7] that when  $t$  becomes of order unity, the interactions become strong, vortex solutions play an important role, effectively renormalizing  $t$ , and a finite correlation length develops. This is the normal phase corresponding to intact chiral symmetry. For the  $U(1)$  model being discussed here,  $t$  can become of order unity only when  $\rho_T/T \rightarrow 0$ , that is, when  $T \rightarrow T_c$  from below. To be more precise, since  $\rho_T \sim (T_c - T)^{1/2}$  in this limit,  $t$  becomes of order unity only

when  $(T_c - T)/T \lesssim 1/N_f$ . But we have already noted that this is the range where the  $1/N_f$  expansion breaks down. The  $\rho$ -field becomes light and must be included along with  $\theta$  in an effective 2D Landau-Ginzburg Lagrangian describing the transition at  $T_c$ , now interpreted to be the transition from the Kosterlitz-Thouless phase to the symmetric phase with finite correlation length. The infrared singularities associated with the 2D description mean that the  $1/N_f$  expansion is not directly useful to describe this transition.

Finally, we comment on the behavior of the thermodynamic free energy  $F(T) = V_{eff}(\rho_T)$  for this continuous-symmetry model, and the proposed inequality constraint [3] using the quantity  $f(T) \equiv -2\pi[F(T) - F(0)]/\zeta(3)T^3$ . In the limit  $T \rightarrow \infty$ , the model is weakly coupled (as was the discrete-symmetry model) with the dynamics governed by a nontrivial UV fixed point of strength  $1/N_f$ . To leading order in  $1/N_f$ ,  $f(T)$  takes the same form as in the discrete-symmetry model:

$$f(T) = -\frac{2N_f}{\zeta(3)T^3} \left( \frac{\rho_T^3}{3} - \frac{\rho_0}{2} \rho_T^2 + \frac{\rho_0^3}{6} + 2T^3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \left( 1 + k \frac{\rho_T}{T} \right) \exp[-k \frac{\rho_T}{T}] \right). \quad (22)$$

Thus to this order,  $f_{UV} = 3N_f$ . There will be  $\mathcal{O}(1)$  corrections to this result due to the weakly interacting UV fixed point.

In the low temperature limit ( $T \ll T_c$ ), the discrete- and continuous- symmetry models behave differently. In both cases, the  $1/N_f$  expansion for the free energy is convergent, describing in the discrete case a mass gap and leading to  $f_{IR} = 0$  to all orders. In the continuous case, there is one massless degree of freedom described by the  $\theta$  field. At momentum scales on the order of  $T$  which determine the free energy, its behaviour is fully 3D, governed by an effective 3D chiral Lagrangian of the form of Eq. 20. Since this 3D Lagrangian is non-interacting,  $f_{IR}$  may be computed exactly to give  $f_{IR} = 1$  corresponding to the single massless degree of freedom. This computation of  $f_{IR}$  is, in effect, an evaluation of the next to leading term in the  $V_{eff}$  given by Eq. 18, for  $T \ll T_c$ . For this model as for the discrete symmetry model,  $f_{IR} < f_{UV}$ .

## 4 Summary and Discussion

We have described the finite temperature phase structure of two 3D NJL models analyzed in a  $1/N_f$  expansion where  $N_f$  is the (large) number of four-component fermions. In the more familiar case of a discrete chiral symmetry, broken at  $T = 0$ , there is a transition from broken symmetry to unbroken symmetry at a non-zero temperature  $T_c$ . The low temperature phase is characterized by a mass gap; since there are no massless degrees of freedom, the quantity  $f_{IR}$  defined in the Section 1 takes the value  $f_{IR} = 0$ . The  $1/N_f$  expansion breaks down as  $T \rightarrow T_c$  due to infrared singularities describable by an effective 2D Landau-Ginzburg theory. At high temperatures, the  $1/N_f$  expansion is again convergent for large  $N_f$  and the theory is governed by a weak  $O(1/N_f)$  UV fixed point. One finds  $f_{UV} = 3 N_f$  up to corrections of  $O(1/N_f)$ .

In the case of a continuous  $U(1) \times U(1)$  symmetry broken at  $T = 0$ , there is no spontaneous breaking of the chiral symmetry at  $T \neq 0$ , although a phase transition still exists at a non-zero temperature  $T_c$ . The normal high-temperature symmetric phase changes at  $T_c$  to a low-temperature phase with its low momentum components ( $p \ll T$ ) described by an effective 2D theory in the Kosterlitz-Thouless (K-T) phase and with a power-law behaviour of correlation functions for  $x \gg 1/T$ . In both finite- $T$  phases, the continuous chiral symmetry is unbroken. The transition from the K-T phase to the broken phase is at  $T = 0$ . There is a single massless degree of freedom for  $T < T_c$  which becomes a conventional Goldstone boson in the limit  $T \rightarrow 0$ . As in the case of discrete symmetry, the  $1/N_f$  expansion breaks down at the phase transition  $T \sim T_c$  due to 2D infrared singularities, but is well behaved away from the transition for large  $N_f$ . We have nothing to say in this paper about the phase structure of NJL models at small  $N_f$  [11].

The quantity  $f_{IR}$  defined in the Section 1 takes the value  $f_{IR} = 1$ , reflecting the presence of a single massless degree of freedom below  $T_c$ . The UV fixed point leads to the value for  $f_{UV} = 3 N_f$  up to corrections of  $O(1/N_f)$  as in the discrete case.

For both models, the exploration of the transition at  $T = T_c$  requires methods that are non-perturbative in  $1/N_f$ . For the discrete case,  $\epsilon$ -expansion methods ( $4 - \epsilon$  dimensions) may be brought to bear although convergence is problematic since  $\epsilon = 2$ . In the continuous case, the conventional  $\epsilon$  expansion is not useful since the model is in the broken phase for  $T < T_c$  for any  $\epsilon < 2$ . In both cases, the breakdown of the  $1/N_f$  expansion is associated with finite-temperature infrared divergences characteristic of a theory in less than four space-time dimensions, and has no direct counterpart in four dimensional theories.

It is interesting to generalize our discussion to NJL models with larger continuous chiral symmetries. Suppose, for example, that the continuous  $U(1) \otimes U(1)$  symmetry of (13) is extended to a  $U(n) \otimes U(n)$  symmetry which at zero temperature breaks to the diagonal  $U(n)$  producing  $n^2$  Goldstone modes. For the case  $n^2 \ll N_f$ , the  $1/N_f$  expansion may be used to analyze the symmetry breaking pattern at zero and finite temperatures. An order by order analysis concludes that for  $T > T_c \sim \rho_0$  (where  $\rho_0$  is a zero temperature vev of an auxilliary field) the model will be in the symmetric phase. As in the abelian case the  $1/N_f$  expansion breaks down when  $T \sim T_c$  due to 2D infrared singularities. For  $T < T_c$ , the fermions are massive, as in the abelian model, and may be integrated out to describe lower energy physics. At momentum scales below  $T$ , an effective 2D chiral Lagrangian emerges, which naturally breaks into two parts. One is the free abelian model we considered previously and the other is an  $SU(n) \times SU(n)$  chiral Lagrangian. The latter is interacting and asymptotically free [6, 12, 13].

The abelian model will be in the KT phase as long as  $T$  is not close to  $T_c$ , ensuring that the coupling strength is small – of order  $1/N_f$ . At intermediate momentum scales,  $p < T$ , the coupling strength of the non-abelian part is also small – of order  $1/N_f$ . The  $\beta$ -function for this coupling  $\tilde{t}$  is  $\beta_{\tilde{t}} \sim -b\tilde{t}^2$  where the constant  $b$  is positive and of order  $n^2/(4\pi)$  for  $n > 1$ . Thus the effective  $\tilde{t}$  coupling runs, becoming of order unity at ultra-low scales  $\mu_{IR} \sim T \exp(-2\pi \cdot N_f/n^2 \cdot \rho_T/T)$ . As in the case of the  $U(1) \otimes U(1)$  model, unit strength is expected to disorder the system leading to a finite correlation length, now of

order  $1/\mu_{IR}$ . But since this happens for any finite  $T < T_c$ , the non-abelian theory is in the ordinary symmetric phase, not the KT phase<sup>3</sup>. For the abelian part only, there is a KT transition as  $T \rightarrow T_c$ . The value of the  $f_{IR}$  is  $n^2$  because although the light scalars of the non-abelian part of the effective theory are massive, their mass  $m_{SU(N)} \sim \mu_{IR}$  vanishes exponentially as  $T \rightarrow 0$ . Thus they are effectively massless in this limit. Clearly, for generic  $n^2 \ll N_f$  the inequality  $f_{IR} \leq f_{UV}$  will always be satisfied since  $f_{UV} = 3N_f$ .

If the  $U(n) \otimes U(n)$  model is arranged so that  $n^2$  becomes of order  $N_f$ , the  $1/N_f$  expansion breaks down at all  $T$  due to the large number of scalar degrees of freedom ( $\mathcal{O}(n^2)$ ) that are formed. It is beyond the scope of this paper to analyze the model when  $n^2 \sim N_f$ , but we note that it may well be accessible to a combined  $1/N_f$ ,  $1/n^2$  expansion. In the limit  $T \ll \rho_T$ , the model is described by an effective, 3D chiral Lagrangian at the scales of  $T$ , and the  $O(N)$  models of this sort have been analyzed using the  $1/N$  expansion [15].

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<sup>3</sup>This conclusion is modified in the presence of the Wess-Zumino-Witten term. However, because the underlying 3D fermionic theory does not carry anomalies, the effective low energy scalar theory is also anomaly free and no WZW term is generated [14].

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